Mathematical model of Synchronization for the Formation of Saturn's Rings & Moons

SURG | Natural Science & Engineering (NSE) | *Tags: Computational/Mathematical Modeling*

This cover page is meant to focus your reading of the sample proposal, summarizing important aspects of proposal writing that the author did well, or could have improved. ***Review the following sections before reading the sample****. The proposal is also annotated throughout to highlight key elements of the proposal's structure and content.*

Proposal Strengths	Areas for Improvement
The proposal does an excellent job breaking down complex mathematical concepts and defining most jargon in simple terms.	A timeline is often useful to help the reader visualize how a project like this fits within the timeframe of the grant (in this case 8 weeks of the summer)
The proposed study has a clear focus and the writer does a good job situating that focus within the broader context of the field using cited evidence. Additionally, the writer points out a clear gap in knowledge and explicitly writes how the methodology will address that gap in knowledge.	The proposal is missing a distinct "preparation" section, which is typically included at the end of the proposal. This section typically outlines the classes, work experience, previous research experience, and technical experience relevant to the project. Additionally the section includes 1-2 sentences showing how this work connects to the researcher's future goals. It is highly suggested to include such a section in your proposal. This author chose to include their preparation and future goals sprinkled throughout the proposal instead, which is discouraged.

Other Key Features to Take Note Of
In mathematics research, it is common to include equations. While it is not always necessary to include these equations within the body of the proposal, it can be extremely helpful. All equations that are crucial to understanding the methodology of your project should be broken down and explained in simple terms. Equations can also be included in an appendix.
Some formatting was lost in the preparation of this sample proposal, which may affect how the components of the equations look within paragraphs.

Starts from broad topic and narrows down to specific issue in simple terms

A definitive explanation for the existence of rings around Saturn has eluded astronomers and physicists for hundreds of years. Why does Saturn have such a robust system of rings, while the other gas giants of Jupiter, Uranus, and Neptune have more thin and faint rings, and the smaller planets don't have rings at all? Popular prevailing theories involve either an ancient moon getting destroyed and spewing its mass around the planet's atmosphere, or the formation of rings from the planetary nebula that created the planet itself (1). A moon could have been ripped apart because the tidal forces from the planets gravity were too great for the moon to withstand, or the moon could have had its outer surface stripped off during the time of the planets formation (3, 4). However, if it is not the case that a destroyed moon is responsible for the formation of Saturn's distinctive rings, and it was purely a process of how the planetary nebula coalesced, why didn't this occur for other planets?

Ideally, by the end of the first paragraph there should be a clear statement about what your project is in broad terms

Identifies gap in knowledge

A major obstacle to confirming these theories is identifying the exact age of the ring system (2). It has been proposed that ring systems are a transient phenomenon, and that eventually all the gas giants will cease to have rings. I want to investigate whether a "synchronization model" can adequately explain why some matter aggregated around Saturn to form moons, why some redistributed into rings, and whether this observation is only temporary. A model of this type could set limits on the minimum and maximum possible ages of the ring system and/or predict when the system will collapse.

Importance of addressing gap

The Kuramoto model, which is a mathematical model for describing the synchronization of systems of coupled oscillators, was first devised by Yoshiki Kuramoto (5). Its applications have included modeling fireflies, lasers, neurons, heart cells, among many other things (6). This model states that the interaction between two oscillators (or in this case, orbiting bodies) depends sinusoidally on their difference in phase. Each oscillator's phase, θi, over time is dependent on its own natural frequency, ωi, the coupling strength between oscillators, K, and the average interaction that each θi has with all other θj:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \qquad i = 1 \ldots N.$$

What is convenient about this model is that it can be solved exactly in the limit $N \to \infty$, something that is quite rare for nonlinear differential equations.

Background goes beyond a summary. Past work is interpreted in terms of what it means for the identified gap in knowledge

Since equation is critical to understanding the project, it is broken down and explained in simple terms

I wish to create a numerical simulation that represents particles orbiting and colliding with each other around Saturn. The goal is to have the collision dynamics lead to radial or angular patterns caused by the tendency of particles to aggregate when they collide. I will then try to extrapolate an effective pairwise interaction potential that is analogous to the Kuramoto model from these observations. I plan to implement this simulation in Matlab, a programming language which I now have almost 3 years of experience from engineering and math related courses and projects. The project will also require knowledge of differential equations, which I have gained from multiple mathematics/engineering courses, as well as an understanding of physical systems, which I have gained from mechanics and astrophysics courses. Ultimately, I want to use this project to acquire research experience and distinguish my graduate school application from others, as I plan to pursue a PhD in applied mathematics after graduation.

Clearly stated aim

Preparation section interrupts methods section instead of coming at the end

The first step of this project is to derive the equations that govern what happens when two particles collide. These include a conservation-of-angular-momentum equation, and a conservation-of-energy equation (with a small fraction of total energy, ε, being lost in each collision). I will also make the simplifying assumption that the particles are orbiting in a perfect

Methods section could benefit from a timeline, however including steps at least shows intended order of methods circle around Saturn. This assumption will allow me to solve the equations for velocity, angular momentum, and energy in terms of only radius and constants. Using these assumptions, conservation of angular momentum is expressed as

$$m_1\sqrt{r_1^i} + m_2\sqrt{r_2^i} = m_1\sqrt{r_1^f} + m_2\sqrt{r_2^f}\,,$$

and conservation of energy (with a small energy loss, ε) is expressed as

$$\frac{m_1}{r_1^f} + \frac{m_2}{r_2^f} = (1-\varepsilon)\left(\frac{m_1}{r_1^i} + \frac{m_2}{r_2^i}\right).$$

Here, $m1$is the mass of the 1st particle, $r1i$ is the initial radius of the 1st particle, and $r^f{}_1$ is the final radius of the 1st particle (that is the radius after a collision has occurred).

Methods go beyond simple procedure. Author explicitly mentions why each step is taken and how that relates to filling the gap in knowledge/specific aim

The next step is to solve these governing equations for $r^f{}_1$ and $r^f{}_2$, so that I can effectively simulate what radius the particles will go to after they collide with each other. I used the computer algebra system Mathematica as an aid when solving these equations. I also solved the simple cases when the two masses were equal and when $\varepsilon = 0$ by hand to verify that the solutions found using Mathematica were correct. When solving the equations, I noticed that there was a restriction on the maximum value of ε that still ensures that angular momentum is conserved. This means that for a collision to conserve angular momentum, it cannot lose an arbitrary amount of energy; the fraction of energy lost must be less than

Shows validation of approach

$$\varepsilon_{max} \le \frac{m_1 m_2\left(r_1^i - r_2^i\right)^2}{\left(m_1 r_1^i + m_2 r_2^i\right)\left(m_2 r_1^i + m_1 r_2^i\right)}.$$

Clear parameters of success

What needs to be done is to create an event-driven simulation in Matlab. This simulation will define a distribution of particles that will orbit Saturn governed by the laws of gravity. The simulation will determine when the next collision between particles will occur, jump to that time, and simulate a collision using the solutions discussed above. Repeating this may result in the particles aggregating or dispersing over time, and I will track the amount of "order" established across the entire system.

Not clear what parameters of success are here.

Once the full-scale simulation with many particles is created, I will try to reconstruct a pairwise interaction potential function from the numerous collisions simulated. Then, I will build and analyze a low-dimensional differential equation based on the effective interaction function. Based on this function, I will be able to assess whether the Kuramoto model is suitable for describing the behavior of this system.

If successful, this model should not only give us insight into the formation of moons vs rings around planetary bodies, but also be applicable to many different scales of astrophysical phenomena. These could include, for example, a better understanding of the differences in the formation of spiral vs elliptical/irregular galaxies formed around supermassive black holes, or the formation of planets vs asteroid belts in heliocentric systems.

The author displays an understanding of what the final output of this research will look like after 8 weeks, and shows how that output relates to answering the research question

Citations:

(1) Elkins-Tanton, L. T. (2011). Jupiter and Saturn. New York: Facts on File

(2) Miner, E. D., Cuzzi, J. N., & Wessen, R. R. (2007). Planetary Ring Systems. New York, NY: Praxis Publishing Ltd, Chichester, UK.

(3) Choi, C. Q. (2010-12-13). "Saturn's Rings Made by Giant "Lost" Moon, Study Hints". National Geographic.

(4) Canup, R. M. (2010-12-12). "Origin of Saturn's rings and inner moons by mass removal from a lost Titan-sized satellite". Nature. 468 (7326): 943–6. Bibcode:2010Natur.468..943C. doi:10.1038/nature09661. PMID 21151108

(5) Kuramoto, Yoshiki (1975). H. Araki, ed. Lecture Notes in Physics, International Symposium on Mathematical Problems in Theoretical Physics. 39. Springer-Verlag, New York. p. 420.

(6) Rodrigues, F. A., Peron, T. K., Ji, P., & Kurths, J. (2016). The Kuramoto model in complex networks. Physics Reports. Retrieved March 13, 2018.